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Fuzzy Set Ideas in Quantum Logics

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Fuzzy set theory language and ideas are used to express basic quantum logic notions. The possibility of replacing probabilistic interpretation of quantum mechanics by interpretation based on infinite-valued logics and fuzzy set theory is outlined. Short review of various structures encountered in the fuzzy set approach to quantum logics is given.

1. INTRODUCTION

Fuzzy set theory was born in 1965 in a paper by Zadeh (1965) but since its very idea can be regarded as a result of applying infinite-valued logic to evaluate the truth-value of a sentence "*x belongs to X*," its roots can be traced back to multiple-valued logics studied by Łukasiewicz (1970) and Post (1921) in the early twenties. Reichenbach (1944) tried to interpret quantum mechanics in terms of a three-valued logic by introducing the third value *indeterminate* besides *true* and *false* to evaluate truth-values of quantum mechanical statements. His ideas, however, were criticized by Feyerabend (1958) as "leading to undesirable consequences." Infinite-valued Łukasiewicz logic L_∞ was considered as a proper propositional calculus for quantum mechanics by Giles (1977). Giles (1976) showed that this logic is related to Zadeh fuzzy sets in the same way as ordinary two-valued logic is related to ordinary sets.

The ideas contained in Giles (1976, 1977) and especially Giles' (1976) notions of bold union and bold intersection are points of departure for the present paper. The other point of departure is the vast field of the quantum logic approach to the foundations of physical theories (Birkhoff and von Neumann, 1936; Mackey, 1963; Jauch, 1968; Piron, 1976; Beltrametti and Cassinelli, 1981). Some of the results of the present paper were developed

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in Pykacz (1987*a*,*b*, 1988, 1989, 1990), but since these papers are difficult to obtain, they are repeated here.

For the convenience of the reader we recall in the next section some basic definitions from the domains of fuzzy set theory and quantum logics.

2. BASIC NOTIONS OF FUZZY SET THEORY AND QUANTUM LOGICS

2.1. Fuzzy Sets

In his historic paper Zadeh (1965) introduced the notion of a fuzzy set in order to describe situations in which certain objects belong to a set "to some extent," contrary to the situation encountered in traditional set theory, where only the cases of complete membership or nonmembership are allowed. In such a way he opened a possibility of studying sets the boundaries of which vanish gradually. Such situations are encountered mainly in "soft" sciences, e.g., psychology, economy, linguistics, medicine, etc., but also in traditional mathematics when we say, for example, "ε *small enough*" or "x *much bigger than* 1," since sets defined by these predicates have no sharp boundaries.

Dubois and Prade (1980) in the Introduction to their book say that there are generally two situations in which fuzziness appears. The first one is encountered when every single object actually belongs or does not belong to a set, i.e., it fully possesses or does not possess a property which distinguishes it from other objects but we do not know which case actually happens. In this case fuzziness appears as a result of lack of our knowledge and this situation usually can be dealt with by probabilistic language and methods. The second situation is a "genuine" fuzzy situation in which fuzziness appears because a predicate which defines a set is vague. Dubois and Prade illustrate the difference between these two situations by applying, respectively, predicates "genuine Indian" and "old" to pieces of pottery in a shop. We shall argue later that in quantum mechanics we encounter still another situation in which fuzziness of a set consisting of pure quantum states is generated by the unavoidable dispersion of experimental results of any test which is used to check a property which defines a set.

Although axiomatic approaches to fuzzy set theory of the type of both Zermelo–Fraenkel (Chapin, 1974, 1975) and Goedel–Bernays (Novak, 1980) axiomatizations have already been given, for our purposes it suffices to adopt Zadeh's original intuitive approach based on the following definitions (Zadeh, 1965; Dubois and Prade, 1980).

Definition 2.1. Let U be a classical set of objects, called a *universe.* A *fuzzy set* $\mathbf{A}$ in U is characterized by a *membership function* $\mu_{\mathbf{A}}: U \to [0, 1]$. The value $\mu_{\mathbf{A}}(x)$ represents the *grade of membership* of x in $\mathbf{A}$.

Definition 2.2. A fuzzy set is *empty* (denoted $\varnothing$) iff its membership function is identically equal to zero on U, i.e., $\mu_{\mathbf{A}}(x)=0$. Two fuzzy sets $\mathbf{A}$ and $\mathbf{B}$ are *equal* iff

$$\mu_{\mathbf{A}}(x)=\mu_{\mathbf{B}}(x) \qquad \text{for all} \quad x\in U \tag{1}$$

and $\mathbf{A}$ is *contained* in $\mathbf{B}$ iff

$$\mu_{\mathbf{A}}(x)\leq\mu_{\mathbf{B}}(x) \qquad \text{for all} \quad x\in U \tag{2}$$

Fuzzy set $\mathbf{A}$ is *normalized* iff there exists $x\in U$ such that $\mu_{\mathbf{A}}(x)=1$.

Remarks. 1. Membership functions are natural generalizations of two-valued characteristic functions of classical sets, therefore classical sets (usually called *crisp* sets in fuzzy literature) are fuzzy sets of a special kind.

2. It was pointed out by Kaufmann (1975) that fuzzy sets are actually generalizations of crisp *subsets* of a universe U. We shall stick, however, to the more popular terminology.

It is possible to endow a family of fuzzy sets with many set-theoretic operations in such a way that after replacing $[0, 1]$-valued membership functions by $\{0, 1\}$-valued characteristic functions, i.e., after coming back to the classical set theory the usual set theoretic union, intersection, and complement are recovered. These operations can be grouped into triples consisting of fuzzy set generalizations of union, intersection, and complement in such a way that the De Morgan laws hold. The most frequently used triple is the triple introduced already by Zadeh in his first paper on fuzzy sets (Zadeh, 1965).

Definition 2.3. Let $\mathbf{A}$ and $\mathbf{B}$ be fuzzy sets with membership functions $\mu_{\mathbf{A}}$ and $\mu_{\mathbf{B}}$, respectively. The (*standard*) fuzzy *union* of $\mathbf{A}$ and $\mathbf{B}$, *intersection* of $\mathbf{A}$ and $\mathbf{B}$, and *complement* of $\mathbf{A}$ are fuzzy sets denoted, respectively, $\mathbf{A}\cup\mathbf{B}$, $\mathbf{A}\cap\mathbf{B}$, and $\mathbf{A}'$, the membership functions of which are given by the following formulas:

$$\mu_{\mathbf{A}\cup\mathbf{B}}(x)=\max[\mu_{\mathbf{A}}(x), \mu_{\mathbf{B}}(x)] \tag{3}$$

$$\mu_{\mathbf{A}\cap\mathbf{B}}(x)=\min[\mu_{\mathbf{A}}(x), \mu_{\mathbf{B}}(x)] \tag{4}$$

$$\mu_{\mathbf{A}'}(x)=1-\mu_{\mathbf{A}}(x) \tag{5}$$

Zadeh operations are generated by Łukasiewicz disjunction, conjunction, and negation in the same way as classical set-theoretic operations are generated by connectives of two-valued logic. For example, the truth-value of the sentence "$x \in \mathbf{A} \cup \mathbf{B}$" equals to the truth-value of the sentence "$x \in \mathbf{A}$ *or* $x \in \mathbf{B}$" both in the case of crisp sets and two-valued logic and in the case of fuzzy sets and infinite-valued Łukasiewicz logic. Łukasiewicz chose negation "$\sim$" and implication "$\rightarrow$" as basic connectives and postulated that

$$\tau(\sim p) = 1 - \tau(p) \tag{6}$$

$$\tau(p \rightarrow q) = \min(1 - \tau(p) + \tau(q), 1) \tag{7}$$

where $\tau(p)$ denotes the truth-value of the sentence p. Max and min expressions for disjunction and conjunction are then obtained by assuming for disjunction the truth-value

$$\tau(p \text{ or } q) = \tau[(p \rightarrow q) \rightarrow q] = \max(\tau(p), \tau(q)) \tag{8}$$

and for conjunction the validity of the De Morgan law

$$\tau(p \text{ and } q) = \tau[\sim(\sim p \text{ or } \sim q)] = \min(\tau(p), \tau(q)) \tag{9}$$

However, as was noticed by Giles (1976), the classical connectives of disjunction and conjunction also have other simple generalization in the infinite-valued Łukasiewicz logic. These other connectives, called by Giles (1976) *bold disjunction* and *bold conjunction*, can be obtained by leaving the formulas (6), (7), and (9) unchanged but changing the relation (8) into the following one, even more familiar and in the case of two-valued logic equivalent:

$$\tau(p \text{ or } q) = \tau(\sim p \rightarrow q) \tag{10}$$

These bold connectives give rise to other set-theoretic operations on fuzzy sets which were defined by Giles (1976) as follows:

Definition 2.4. Let $\mathbf{A}$ and $\mathbf{B}$ be fuzzy sets. *Bold union* (denoted $\mathbf{A} \cup \mathbf{B}$) and *bold intersection* (denoted $\mathbf{A} \cap \mathbf{B}$) are fuzzy sets with membership functions

$$\mu_{\mathbf{A} \cup \mathbf{B}}(x) = \min(\mu_{\mathbf{A}}(x) + \mu_{\mathbf{B}}(x), 1) \tag{11}$$

$$\mu_{\mathbf{A} \cap \mathbf{B}}(x) = \max(\mu_{\mathbf{A}}(x) + \mu_{\mathbf{B}}(x) - 1, 0) \tag{12}$$

If $\mathbf{A} \cap \mathbf{B} = \varnothing$, then $\mathbf{A}$ and $\mathbf{B}$ are called *weakly disjoint.*

Let us notice that membership functions of weakly disjoint sets satisfy the following inequality:

$$\mu_{\mathbf{A}}(x) + \mu_{\mathbf{B}}(x) \leq 1 \qquad \text{for all} \quad x \in U \tag{13}$$

We shall argue later that bold operations are better for describing properties of quantum mechanical systems than standard fuzzy operations. Let us note here one fact which indicates this direction. The family $\mathbb{F}(U)$ of all fuzzy subsets of a given universe U endowed with standard fuzzy operations is a De Morgan lattice, i.e., a distributive lattice with standard fuzzy complement as involution:

$$\mathbf{A}'' = \mathbf{A} \tag{14}$$

This lattice is partially ordered by the fuzzy set inclusion (2). The standard fuzzy union and intersection are, respectively, the least upper bound and the greatest lower bound with respect to this partial order. However, the standard fuzzy complement is not a complement in the lattice-theoretic sense since neither the excluded-middle law nor the law of contradiction hold for a genuine (i.e., noncrisp) fuzzy set $\mathbf{A}$:

$$\mathbf{A} \cup \mathbf{A}' \neq U, \qquad \mathbf{A} \cap \mathbf{A}' \neq \varnothing \tag{15}$$

This is no longer true for Giles bold union and intersection combined with the standard fuzzy complement:

$$\mathbf{A} \boldsymbol{\cup} \mathbf{A}' = U, \qquad \mathbf{A} \boldsymbol{\cap} \mathbf{A}' = \varnothing \tag{16}$$

Moreover, contrary to the De Morgan lattice $(\mathbb{F}(U), \cup, \cap, ')$, the structure $(\mathbb{F}(U), \boldsymbol{\cup}, \boldsymbol{\cap}, ')$ is nondistributive and therefore it is more similar to structures encountered in the quantum logic approach.

2.2. Quantum Logics

We now recall definitions of basic quantum logic notions which will be used in the sequel. The reader interested in their physical justification is referred to the books of Mackey (1963), Jauch (1968) Piron (1976), or Beltrametti and Cassinelli (1981) which is a real encyclopedia of the quantum logic approach.

Definition 2.5. A *quantum logic* (or simply a *logic*) is an orthomodular σ-orthocomplete orthoposet, i.e., a partially ordered set L which contains the smallest element $\mathbf{0}$ and the greatest element $\mathbf{I}$, in which the orthocomplementation map $' : L \to L$ satisfying the conditions (i)–(iii) exists:

(i) $a'' = a$

(ii) if $a \leq b$, then $b \leq a$

(iii) the greatest lower bound (*meet*) $a \wedge a'$ and the least upper bound (*join*) $a \vee a'$ exist in L and $a \wedge a' = \mathbf{0}$, $a \vee a' = \mathbf{I}$

and the σ-orthocompleteness condition

(iv) if $a_i \leq a_j'$ for $i \neq j$, then the join $\bigvee_i a_i$ exists in L

and orthomodular identity

(v) if $a \leq b$, then $b = a \vee (b \wedge a')$

hold.

Let us note that the existence of the meet $b \wedge a'$ and the join $a \vee (b \wedge a')$ in the right-hand side of an orthomodular identity follows from the σ-orthocompleteness condition and the De Morgan laws, which state that meet and join, when they exist, are not independent:

$$(a \vee b)' = a' \wedge b', \qquad (a \wedge b)' = a' \vee b' \tag{17}$$

Elements which satisfy assumptions of the σ-orthocompleteness condition i.e., such that $a_i \leq a_j'$ for $i \neq j$, are usually called *orthogonal* (or *disjoint*) and denoted $a \perp b$. Therefore, the σ-orthocompleteness condition consists in assuming that the join of every countable pairwise orthogonal sequence exists in L.

Definition 2.6. A *probability measure* (*state*) on a logic L is a map $p: L \to [0, 1]$ such that $p(\mathrm{I}) = 1$ and p is σ-additive for every countable pairwise orthogonal sequence, i.e., if $a_i \perp a_j$ for $i \neq j$, then the series $\sum_i p(a_i)$ converges and

$$p\left(\bigvee_i a_i\right) = \sum_i p(a_i) \tag{18}$$

Since $a \leq a = (a')'$, the two-element sequence $\{a, a'\}$ is orthogonal and we obtain immediately $p(a) + p(a') = p(a \vee a') = p(\mathrm{I}) = 1$.

If $p(a) = p(b)$ for all p belonging to some set of states implies $a = b$, then this set of states is called *ordering* (Beltrametti and Cassinelli, 1981) or *full* (Mackey, 1963). It is generally assumed in the quantum logic approach to any physical theory that the set of all states S on a logic L is ordering, i.e., that

$$p(a) = p(b) \qquad \text{for all} \quad p \in S \quad \text{implies} \quad a = b \tag{19}$$

One can easily check that if $\{p_i\}$ is any collection of states on a logic L and $\{w_i\}$ is any collection of real numbers such that $0 \leq w_i \leq 1$ and $\sum_i w_i = 1$, then a convex combination $\sum_i w_i p_i$ defined in a pointwise manner

$$\left(\sum_i w_i p_i\right)(a) = \sum_i w_i p_i(a) \tag{20}$$

is again a state on a logic L (in the case of infinite sequences, the sums should be understood as limits of suitable finite sums).

Definition 2.7. A state p on a logic L is *pure* if it cannot be represented in the form of a convex combination of other states. A state which is not pure is called *mixed.*

Definition 2.8. A state p on a logic L is *dispersion-free* if for any $a \in L$ either $p(a) = 0$ or $p(a) = 1$.

It is usually argued that elements of a logic, usually called *propositions*, represent the most primitive statements about physical systems, i.e., statements which are confirmed or falsified in every single run of any experiment designed to check them. The name "state" given to a probability measure on a logic is not accidental—it is assumed that it actually represents a state of a physical system and that pure (mixed) states of a physical system are represented, respectively, by pure (mixed) states on a logic. The link with experiments is established by the fundamental assumption of most quantum logic approaches (Birkhoff and von Neumann, 1936; Mackey, 1963; Jauch, 1968; Piron, 1976; Beltrametti and Cassinelli, 1981) that the number $p(a)$ is the probability of obtaining positive (yes) result for the proposition "a" when the physical system is in the state represented by "p." Since knowledge about the structure of the set of propositions is assumed to come from experiments performed on a physical system prepared to be in different states, the assumption that the set of all states is ordering seems to be unavoidable.

There are two standard examples of logics of physical systems:

1. A lattice of closed subspaces of a Hilbert space in quantum mechanics.
2. A Boolean algebra of subsets of a phase space in classical mechanics.

Let us recall that a lattice is a poset in which meet and join of any two elements exist and a Boolean algebra is an orthocomplemented lattice in which every triple of elements (a, b, c) is distributive, i.e.,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{21}$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{22}$$

Our considerations are more general since we have not assumed a logic to be a lattice, which means that meets and joins of arbitrary elements do not always have to exist.

3. FUZZY QUANTUM LOGICS

Since for any element a of a logic L and for any state p the number $p(a)$ belongs to the unit interval, states can be treated as fuzzy subsets of a universe L, and conversely, propositions can be treated as fuzzy subsets of a universe S. This second possibility allows us to utilize the fuzzy set theory with the aid of the following theorem of Mączyński (1973, 1974).

Theorem 3.1. (i) If L is a logic with an ordering set of probability measures S, then each $a \in L$ induces a function $\underset{\sim}{a} : L \rightarrow [0, 1]$,where $\underset{\sim}{a}(p) = p(a)$ for all $p \in S$. The set of all such functions $\underset{\sim}{L} = \{\underset{\sim}{a} : a \in L\}$ satisfies the following condition.

Orthogonality Postulate: If $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots$ is a sequence of functions such that $\underset{\sim}{a}_i + \underset{\sim}{a}_j \leq 1$ for $i \neq j$, then there exists $\underset{\sim}{b} \in \underset{\sim}{L}$ such that $\underset{\sim}{b} + \underset{\sim}{a}_1 + \underset{\sim}{a}_2 + \cdots = 1$.
$\underset{\sim}{L}$ equipped with the natural partial order $\underset{\sim}{a} \leq \underset{\sim}{b}$ iff $\underset{\sim}{a}(p) \leq \underset{\sim}{b}(p)$ for all $p \in S$ and complementation $\underset{\sim}{a}' = 1 - \underset{\sim}{a}$ is isomorphic to L.

(ii) Conversely, if $\underset{\sim}{L}$ is a set of functions from X into [0, 1] for which the orthogonality postulate is satisfied, then it is a logic with respect to the natural partial order and complementation. Every point $x \in X$ induces a probability measure m_x on $\underset{\sim}{L}$ where $m_x(\underset{\sim}{a}) = \underset{\sim}{a}(x)$ for all $\underset{\sim}{a} \in \underset{\sim}{L}$ and the set of all such measures $\{m_x : x \in X\}$ is ordering.

Thanks to this theorem we see that any logic L with an ordering set of states S is isomorphic to a family $\mathbb{L}$ of fuzzy subsets of S equipped with the standard fuzzy set inclusion and complementation, and such that membership functions of elements of $\mathbb{L}$ satisfy the orthogonality postulate. Such a view on the logics of physical systems was proposed in Pykacz (1987*a*). According to it the number $p(a)$, instead of being interpreted as the probability of obtaining a positive result in an experiment testing the property a when a physical system Ω is in the state p, is interpreted as follows:

$p(a) = \mu_{\mathbf{A}}(p)$ is the grade of membership of a state p to the (fuzzy) subset $\mathbf{A}$ of the set of all states S of the physical system Ω. The subset $\mathbf{A}$ is defined by the predicate:

"the result of an experiment testing the property a is positive"

or by the predicate:

"the physical system Ω has the property a"

Remark. We would like to stress that, according to this point of view, one is perfectly allowed to say that a physical system Ω prepared to be in a state p has a property a even before this property is measured. Before the measurement the sentence "q" = "*the system Ω in the state p has the property a*" should be understood as belonging to the domain of infinite-valued logic and its truth-value $\tau(q)$ can be any number from interval [0, 1]. Of course at the same time the sentence "$\sim q$" = "*the system Ω in the state p does not have the property a*" is also meaningful and $\tau(\sim q) = 1 - \tau(q)$. According to the very idea of the fuzzy set theory the number $\tau(q)=p(a)$ can be also interpreted as the degree to which the physical system Ω in the state p has the property a.

Since membership functions of weakly disjoint sets $\mathbf{A}\cap\mathbf{B}=\varnothing$, according to formula (13), satisfy the condition $\mu_{\mathbf{A}}+\mu_{\mathbf{B}}\leq 1$, the orthogonality postulate can be translated into the fuzzy set language in the following way (Pykacz, 1987*a*, 1988, 1989):

Definition 3.1. We say that in the family $\mathbb{F}$ of fuzzy sets the *fuzzy orthogonality postulate* is satisfied if for any sequence $\mathbf{A}_1, \mathbf{A}_2, \ldots$ of pairwise weakly disjoint sets $\sum_i \mu_{\mathbf{A}_i}\leq 1$ and there exists $\mathbf{B}\in\mathbb{F}$ such that

$$\mathbf{B}=\left(\bigcup_i \mathbf{A}_i\right)' \tag{23}$$

Let us note that if the fuzzy orthogonality postulate is satisfied and $\mathbf{A}_1, \mathbf{A}_2, \ldots$ are pairwise weakly disjoint, then

$$\mu_{\cup_i \mathbf{A}_i}=\sum_i \mu_{\mathbf{A}_i} \tag{24}$$

By simply translating Mączyński's (1973, 1974) results into the language of fuzzy sets, we can check that the fuzzy orthogonality postulate is satisfied in $\mathbb{F}$ if and only if in $\mathbb{F}$ the following conditions are satisfied:

(1.1) $\mathbb{F}$ contains the empty set $\varnothing$

(1.2) $\mathbb{F}$ is closed under the standard fuzzy set complementation

(1.3) If $\mathbf{A}_1, \mathbf{A}_2, \ldots$ are pairwise weakly disjoint, then $\sum_i \mu_{\mathbf{A}_i}\leq 1$ and $\bigcup_i \mathbf{A}_i\in\mathbb{F}$.

Definition 3.2. By a *fuzzy quantum logic* we mean any family of fuzzy sets in which the fuzzy orthogonality postulate or, equivalently, conditions (1.1)–(1.3) are satisfied.

Throughout the rest of the paper a fuzzy quantum logic consisting of fuzzy subsets of a universe U will be denoted $\mathbb{L}(U)$.

By part (ii) of the Mączyński theorem any fuzzy quantum logic $\mathbb{L}(U)$ is a traditional quantum logic in which partial order coincides with the standard fuzzy set inclusion and orthocomplementation is the standard fuzzy set complementation. The whole universe U and the empty set $\varnothing$ are, respectively, the greatest and the least elements of a fuzzy quantum logic $\mathbb{L}(U)$. Conversely, by part (i) of this theorem, any traditional quantum logic with an ordering set of probability measures S is isomorphic to a fuzzy quantum logic $\mathbb{L}(S)$.

We can express the definition of a probability measure (state) on a fuzzy quantum logic in fuzzy set terms in the following way:

Definition 3.3. By a *probability measure* (*state*) on a fuzzy quantum logic $\mathbb{L}(U)$ we mean a mapping $p: \mathbb{L}(U) \to [0, 1]$ such that $p(U) = 1$ and

$$p\left(\bigcup_i \mathbf{A}_i\right) = \sum_i p(\mathbf{A}_i) \tag{25}$$

for any sequence of weakly disjoint sets.

Let us note that by the very definition and by the Mączyński theorem any fuzzy quantum logic $\mathbb{L}(U)$ admits an ordering set of probability measures induced by points $x \in U$ with the aid of the following formula:

$$p_x(\mathbf{A}) = \mu_{\mathbf{A}}(x) \qquad \text{for all} \quad \mathbf{A} \in \mathbb{L}(U) \tag{26}$$

However, generally there can exist probability measures on a fuzzy quantum logic which are not induced by points of the universe U. For example, if all membership functions of elements of a fuzzy quantum logic $\mathbb{L}(U)$ are integrable on the set U and if we define

$$p(\mathbf{A}) = c \int_U \mu_{\mathbf{A}}(x)\, dx, \qquad \text{where} \quad c = \left(\int_U dx\right)^{-1} \tag{27}$$

then $p(U) = 1$ and from the formula (24) it follows that

$$p\left(\bigcup_i \mathbf{A}_i\right) = c \int_U \left(\sum_i \mu_{\mathbf{A}_i}(x)\right) dx = \sum_i p(\mathbf{A}_i) \tag{28}$$

for any sequence of pairwise weakly disjoint sets. Therefore $p: \mathbb{L}(U) \to [0, 1]$ is a probability measure on $\mathbb{L}(U)$.

Example 3.1. The most standard example of a fuzzy quantum logic can be obtained via the Mączyński theorem from the traditional quantum logic of projectors on a Hilbert space H. Let $P(H)$ be such a logic and let $S(H)$ denote the set of all density matrices on H. The family $\mathbb{L}(S(H))$ of all fuzzy subsets of $S(H)$, the membership functions of which are defined by

$$\mu_{\mathbf{P}}(\rho) = \mathrm{Tr}(\rho \mathbf{P}) \qquad \text{for all} \quad \rho \in S(H) \tag{29}$$

where $\mathbf{P} \in \mathbb{L}(S(H))$ denotes the fuzzy subset of $S(H)$ generated by the projector $P \in P(H)$, is a fuzzy quantum logic isomorphic to $P(H)$. Probability measures on $\mathbb{L}(S(H))$ generated by density matrices are of the form

$$m_\rho: \quad \mathbb{L}(S(H)) \to [0, 1], \qquad m_\rho(\mathbf{P}) = \mathrm{Tr}(\rho P) \tag{30}$$

Example 3.2. Let X be a topological space and let $\mathbb{B}(X)$ be a Boolean algebra of Borel subsets of X. $\mathbb{B}(X)$ is a fuzzy quantum logic in which all elements are crisp and all traditional probability measures on $\mathbb{B}(X)$ are states in the sense of the Definition 3.2.

Before we pass to the comparison of fuzzy quantum logics with other structures encountered in the fuzzy set approach to quantum logics, let us introduce, after Piasecki (1985), the following definition:

Definition 3.4. Any fuzzy set $\mathbf{E}$ such that

$$\mu_{\mathbf{E}} \leq \mu_{\mathbf{E}'} \tag{31}$$

is called a *weakly empty* set. Any fuzzy set $\mathbf{U}$ such that

$$\mu_{\mathbf{U}'} \leq \mu_{\mathbf{U}} \tag{32}$$

is called a *weak universe*.

The pathological behavior of elements of quantum logics such that $a \leq a'$ for $a \neq \mathbf{0}$ or $a' \leq a$ for $a \neq \mathbf{I}$ was recognized long ago. For example, in Mackey (1963) the very existence of such elements is excluded by Mackey's Axiom VIII. The structure of a "good" quantum logic should guarantee the nonexistence of elements described above. The orthogonality postulate or its fuzzy version is strong enough to generate such a structure, so the following theorem is strongly indicated.

Theorem 3.2. A fuzzy quantum logic $\mathbb{L}(U)$ does not contain any weakly empty set and any weak universe except $\varnothing$ and U.

Proof. Let $\mathbf{E}$ be any weakly empty set and let $\mathbf{U}$ be any weak universe in a fuzzy quantum logic $\mathbb{L}(U)$. Formula (31) implies that $\mathbf{E} \wedge \mathbf{E}' = \mathbf{E}$ and formula (32) implies that $\mathbf{U} \vee \mathbf{U}' = \mathbf{U}$, where $\wedge$ and $\vee$ denote, respectively, meet and join in $\mathbb{L}(U)$ partially ordered by the standard fuzzy set inclusion. By part (ii) of the Mączyñski theorem the standard fuzzy set complementation in $\mathbb{L}(U)$ is an orthocomplementation. Therefore, the condition (iii) of Definition 2.5 implies that $\mathbf{E} = \varnothing$ and $\mathbf{U} = \mathrm{U}$. ■

Corollary 3.1. A fuzzy quantum logic $\mathbb{L}(U)$ does not contain any set whose membership function is constant, except crisp sets $\varnothing$ and U.

This result, from the physical point of view, is quite natural, since if $\mathbb{L}(S)$ is a logic of a physical system, and if $\mu_{\mathbf{A}}$ were a constant function on the set of states S, then $\mu_{\mathbf{A}}$ would give no information about the structure of S (which reflects features of a physical system) and therefore would be quite useless. The crisp sets U and $\varnothing$ which belong to any fuzzy quantum logic $\mathbb{L}(S)$ can be thought of as representing trivial properties of existence and nonexistence of a physical system and they (or their nonfuzzy counterparts $\mathbf{I}$ and $\mathbf{0}$) are added to a logic of a physical system mainly for mathematical convenience.

The fact that a fuzzy quantum logic does not contain any noncrisp set whose membership function is constant can be inferred also from the following theorem:

Theorem 3.3. If a fuzzy quantum logic $\mathbb{L}(U)$ contains a set $\mathbf{A} \neq \varnothing$ with memberships function $\mu_{\mathbf{A}}$ then it does not contain any set whose membership function equals $c\mu_{\mathbf{A}}$ for $c \in (0, 1)$.

We shall prove first the following lemma, which is interesting in its own right.

Lemma 3.1. If $\mathbf{A}$ and $\mathbf{B}$ are two elements of a fuzzy quantum logic $\mathbb{L}(U)$ and $\mathbf{A}$ is contained in $\mathbf{B}$, i.e., $\mu_{\mathbf{A}} \leq \mu_{\mathbf{B}}$, then there exists $\mathbf{C} \in \mathbb{L}(U)$ such that $\mu_{\mathbf{C}} = \mu_{\mathbf{B}} - \mu_{\mathbf{A}}$.

Proof. If $\mathbf{A}$ is contained in $\mathbf{B}$, then $\mu_{\mathbf{A}} + 1 - \mu_{\mathbf{B}} = \mu_{\mathbf{A}} + \mu_{\mathbf{B}'} \leq 1$, which means that $\mathbf{A}$ and $\mathbf{B}'$ are weakly disjoint. Therefore, by the fuzzy orthogonality postulate, there exists $\mathbf{C} \in \mathbb{L}(U)$ such that $\mu_{\mathbf{C}} = 1 - (\mu_{\mathbf{A}} + \mu_{\mathbf{B}'}) = \mu_{\mathbf{B}} - \mu_{\mathbf{A}}$. ■

Proof of Theorem 3.3. Let us denote by $c\mathbf{A}$ the fuzzy set whose membership function equals $c\mu_{\mathbf{A}}$. If $c \in (0, 1/2)$, then $c\mathbf{A}$ is a nonempty, weakly empty set and its existence in $\mathbb{L}(U)$ is excluded by Theorem 3.2.

Let us suppose now that $c \in [1/2, 1)$. If $c\mathbf{A}$ belonged to $\mathbb{L}(U)$, then, by Lemma 3.1, $(1-c)\mathbf{A}$ with the membership function $(1-)\mu_{\mathbf{A}} = \mu_{\mathbf{A}} - c\mu_{\mathbf{A}}$ would belong to $\mathbb{L}(U)$ as well. Since $1 - c \in (0, 1/2]$, $(1-c)\mathbf{A}$ is a nonempty, weakly empty set which cannot belong to $\mathbb{L}(U)$ by Theorem 3.2. ■

Again the fact stated in Theorem 3.3 is not surprising from the physical point of view. If we had two elements $\mathbf{A}$ and $c\mathbf{A}$ of a fuzzy quantum logic, both of them could be treated as representing the same property a of a physical system. For example, let us study a property a = "*incoming particles are linearly polarized in the direction z*" with the aid of a measuring device M_1 consisting of a linear polarizer LP_z oriented in the direction z and a counter C_1 placed behind it. By applying various preparation procedures to incoming particles, i.e., by preparing the particles to be in different states, we can obtain, at least approximately, the membership function which characterizes the set of states $\mathbf{A}$ defined by the property a. Now let us replace the measuring device M_1 by a new one M_2 in which the counter C_1 is replaced by a counter C_2 whose sensitivity is 80% of the previous one. All experimental outcomes are diminished now by the factor 0.8, but this change does not reflect any change of the studied property, but only a change in a measuring device. Of course it could be argued that with the aid of M_1 and M_2 we study two different properties: a_1 = "*particles pass through LP_z and are detected by C_1*" and a_2 = "*particles pass through LP_z and are detected by C_2*," but it is obvious that by studying property a_2 we do not obtain any information about a physical system which could not be obtained by studying a_1. Therefore, there is no reason to include both properties in the mathematical description of the physical system at least for the sake of nonredundancy.

4. OTHER STRUCTURES ENCOUNTERED IN THE FUZZY SET APPROACH TO QUANTUM LOGICS

This section is devoted to the comparison of three notions that are different from ours and were independently introduced in the attempt at utilizing fuzzy set ideas in the domain of quantum logic. To make this comparison easier, we shall rewrite all relevant definitions using the notation adopted in the present paper.

4.1. Fuzzy σ-Orthoposets

The notion of a fuzzy σ-orthoposet was introduced by Guz (1984) in the following way.

Definition 4.1. A fuzzy σ-orthoposet is a family $\mathbb{G}$ of fuzzy subsets of a universe U such that:

(o.1) $\mathbb{G}$ contains the empty set $\varnothing$ and the universe U.

(o.2) If $\mathbf{A}$, $\mathbf{B}\in\mathbb{G}$ and $\mathbf{B}$ contains $\mathbf{A}$, then there exists $\mathbf{C}\in\mathbb{G}$ such that $\mu_{\mathbf{C}}=\mu_{\mathbf{B}}-\mu_{\mathbf{A}}$.

(o.3) For every sequence $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \ldots\}$ in $\mathbb{G}$ such that $\sum_i \mu_{\mathbf{A}i}\leq 1$, there exists $\mathbf{B}$ in $\mathbb{G}$ such that $\mu_{\mathbf{B}}=\sum_i \mu_{\mathbf{A}i}$.

One can easily notice that the notion of a fuzzy σ-orthoposet is more general than the notion of a fuzzy quantum logic. Indeed, if $\mathbb{L}$ is a fuzzy quantum logic, then the conditions (1.1) and (1.2) imply the condition (o.1). It follows from Lemma 3.1 that the condition (o.2) is also satisfied and since any sequence described in the condition (o.3) is pairwise weakly disjoint, from (24) and (1.3) it follows that (o.3) is satisfied as well. Thus, we have proved the following result.

Theorem 4.1. Any fuzzy quantum logic is a fuzzy σ-orthoposet.

Guz (1984) gave several examples of fuzzy σ-orthoposets. The following two are the most interesting from the physical point of view.

Example 4.1. The Boolean σ-algebra of ordinary (crisp) sets.

Example 4.2. The family of fuzzy subsets of a complex Hilbert space H whose membership functions are generated by the C^*-algebra $B(H)$ of bounded operators acting on H in the following way:

$$\mu_{\mathbf{A}}(x) = (Ax, x)/\|x\| \quad \text{if } x \neq 0$$
$$\mu_{\mathbf{A}}(0) = 0, \qquad \text{for all} \quad x \in H, \quad A \in B(H) \tag{33}$$

It can be easily seen, for instance by comparing these two examples, respectively, with Examples 3.2 and 3.1, that both the above-mentioned fuzzy σ-orthoposets are fuzzy quantum logics. This is not the case, because of Theorems 3.2 and 3.3 and Corollary 3.1 in the following example.

Example 4.3. The family $\mathbb{F}(U)$ of all fuzzy subsets of a fixed crisp set U.

However, the comments at the end of Section 3 indicate that the physical significance of the structure described in Example 4.3 should not be expected to be very big. Therefore, we think that the notion of a fuzzy σ-orthoposet is too general to efficiently describe situations encountered in physics.

Before we close this section let us comment upon the following notions introduced in Guz (1984).

Definition 4.2. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two fuzzy σ-orthoposets. A map $h: \mathbb{G}_1 \to \mathbb{G}_2$ is said to be a *σ-homomorphism* under the following conditions.

(i) When $\mathbf{A}$ is contained in $\mathbf{B}$, then $h(\mathbf{A})$ is contained in $h(\mathbf{B})$.

(ii) For every sequence $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \ldots\}$ in $\mathbb{G}_1$ which satisfies condition (o.3) of Definition 4.2, the sequence of images $\{h(\mathbf{A}_1), h(\mathbf{A}_2), \ldots, h(\mathbf{A}_n), \ldots\}$ in $\mathbb{G}_2$ also satisfies (o.3) and

$$\mu_{h(\mathbf{B})} = \sum_i \mu_{h(\mathbf{A}_i)} \tag{34}$$

(iii) If, moreover, $\mathbb{G}_1$ and $\mathbb{G}_2$ consist of fuzzy subsets of the same universe U and h does not diminish grades of membership, i.e.,

$$\mu_{\mathbf{A}} \leq \mu_{h(\mathbf{A})} \qquad \text{for all} \quad \mathbf{A} \in \mathbb{G}_1 \tag{35}$$

then the homomorphism h is called *proper*.

Definition 4.3. If $\mathbb{G}_1$ is a Boolean σ-algebra and $h: \mathbb{G}_1 \to \mathbb{G}_2$ is a proper homomorphism, then the pair $(\mathbb{G}_2, h)$ is called a *fuzzy extension* of the Boolean σ-algebra $\mathbb{G}_1$.

It can be checked that in fact any proper homomorphism is an identity mapping. These explains why, as observed in (Guz, 1984) there is no nontrivial fuzzy extension for any Boolean σ-algebra consisting exclusively of crisp sets.

4.2. Fuzzy σ-algebras

Contrary to fuzzy σ-orthoposets, fuzzy σ-algebras studied by Guz (1985) are more specific objects than fuzzy quantum logics. The definition of a fuzzy σ-algebra translated into the language used throughout the present paper is the following:

Definition 4.4. A *fuzzy σ-algebra* [also called by Guz (1985) *statistical σ-algebra*] is a family $\mathbb{S}$ of fuzzy subsets of a universe U which satisfies the following axioms (a.1)–(a.7):

(a.1) If $\mu_{\mathbf{A}}(x)=\mu_{\mathbf{B}}(x)$ for all $x\in U$, then $\mathbf{A}=\mathbf{B}$.
(a.2) $\mathbb{S}$ contains the universe U.
(a.3) $\mathbb{S}$ is closed under the standard fuzzy complementation.
(a.4) For any sequence $\{\mathbf{A}_i\}$ of pairwise weakly disjoint elements of $\mathbb{S}$ there is an element $\mathbf{B}$ of $\mathbb{S}$ such that

$$\mu_{\mathbf{B}}(x)=\sum_i \mu_{\mathbf{A}_i}(x) \qquad \text{for all} \quad x\in U \tag{36}$$

(a.5) If $\mathbf{A}, \mathbf{B}\in\mathbb{S}$ are not weakly disjoint, then there is $x\in U$ such that

$$\mu_{\mathbf{A}}(x)=1 \qquad \text{and} \qquad \mu_{\mathbf{B}}(x)>0 \tag{37}$$

(a.6) For each $x\in U$ there is $\mathbf{A}\in\mathbb{S}$ such that

$$\mu_{\mathbf{A}}(x)=1 \qquad \text{and} \qquad \mu_{\mathbf{A}}(y)<1 \qquad \text{for all} \quad y\in U, \quad y\neq x \tag{38}$$

(a.7) If $\mu_{\mathbf{A}}(x)>0$, where $x\in U$ and $\mathbf{A}\in U$, then there exists one and only one point $y\in U$ such that $\mu_{\mathbf{A}}(y)=1$ and $\mu_{\mathbf{A}}(x)=(x:y)$, where $(x:y)$, the so-called *transition probability* from x to y, is defined by

$$(x:y)=\inf\{\mu_{\mathbf{A}}(x)\colon \mathbf{A}\in\mathbb{S},\ \mu_{\mathbf{A}}(y)=1\} \tag{39}$$

It is easy to notice that the Guz axioms (a.2)–(a.4) are equivalent to conditions (1.1)–(1.3) which define a fuzzy quantum logic; therefore, we obtain the following result.

Theorem 4.2. Any fuzzy σ-algebra is a fuzzy quantum logic.

However, other Guz axioms [besides axiom (a.1), which is nothing more than the definition of equality of fuzzy sets] make fuzzy σ-algebras a proper subclass of a class of fuzzy quantum logics. It can be checked that some results of Guz (1985), like the theorem which says that any fuzzy σ-algebra is an orthomodular σ-orthoposet with the least and the greatest element, depend only on axioms (a.1)–(a.4) [in fact this theorem is a "fuzzy" version of Theorem 3.1 proved by Mączyñski (1973)]. Other results, e.g., that any fuzzy σ-algebra is an atomistic σ-orthoposet satisfying the covering law, depend on all Guz axioms, so they could not be obtained in the general fuzzy quantum logic framework.

4.3. Fuzzy Quantum Spaces

The notion of a fuzzy quantum space was introduced by Dvurečenskij and Chovanec (1988) as a fuzzy generalization of a quantum space of Suppes (1966). This notion also shows remarkable similarities to the notion of a fuzzy quantum logic.

Definition 4.5. A *fuzzy quantum space* is a family $\mathbb{M}$ of fuzzy subsets of a universe U such that:

(s.1) $\mathbb{M}$ contains the empty set $\varnothing$.
(s.2) $\mathbb{M}$ is closed under the standard fuzzy set complementation.
(s.3) If $\mu_{\mathbf{A}}(x)=1/2$ for all $x\in U$, then $\mathbf{A}\notin\mathbb{M}$.
(s.4) If $\mathbf{A}_1, \mathbf{A}_2, \ldots$ are pairwise weakly disjoint, then $\bigcup_i \mathbf{A}_i\in\mathbb{M}$.

When we compare Definition 4.5 with Definition 3.2 of a fuzzy quantum logic we see that conditions (s.1) and (1.1), as well as conditions (s.2) and (1.2), are identical. However, the union in the condition (s.4) is the standard (Zadeh) fuzzy set union whose membership function is defined as the pointwise supremum of membership functions of sets $\mathbf{A}_i$:

$$\mu_{\bigcup_i \mathbf{A}_i}(x)=\sup_i(\mu_{\mathbf{A}_i}(x)) \tag{40}$$

while in the analogous condition (1.3) of the definition of a fuzzy quantum logic, Giles' bold union is utilized. Moreover, according to the condition (1.3) of Definition 3.2 the algebraic sum of membership functions of pairwise weakly disjoint sets should not exceed 1. This is a very restrictive condition and mainly due to this condition, fuzzy quantum logics have the sophisticated structure of an orthocomplemented orthomodular set.

Let us note that in the definition of a fuzzy quantum space $\mathbb{M}$ of Dvurečenskij and Chovanec there is nothing which could prevent $\mathbb{M}$ from containing a noncrisp, weakly empty set, weak universe, or sets of the form $c\mathbf{A}$, which explains why in the condition (s.3) of Definition 4.5 nonexistence of a fuzzy set with membership function constantly equal to 1/2 had to be assumed separately. Nevertheless, this condition was adopted by Dvurečenskij and Chovenec because of mathematical convenience (in fact their paper is mainly a mathematical one and it does not contain any application of their notions in theoretical or experimental physics).

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